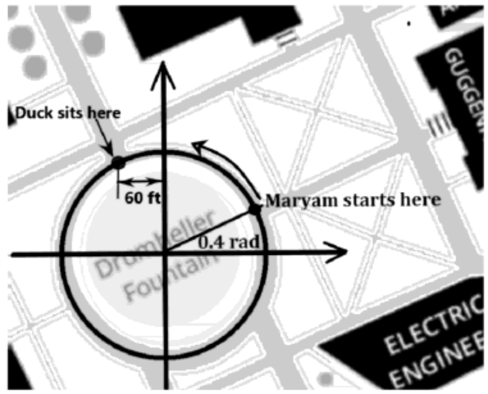
SOLUTIONS

1. Maryam gets out of the Math 120 final in Guggenheim and, to celebrate, she starts running counterclockwise on the path around the Drumheller fountain, which has a circumference of 660 feet. It takes her 55 seconds to run a full victory lap (one time around the fountain), at constant speed.

   (a) (8 points) Impose a coordinate system with origin at the center of the fountain. Maryam's starting position is as indicated. Compute her (x, y)-coordinates after 38 seconds of running.



- circumference $= 2\pi R = 660$ feet
  $\Rightarrow R = \frac{660}{2\pi} = \frac{330}{\pi} \cong 105.0422$ ft.
- angular velocity is positive (counter-clockwise) and $\omega = \frac{2\pi \text{ rad}}{55 \text{ sec}}$
- Coordinates at $t = 38$ sec. are:

$$x = R\cos(\theta_0 + \omega t)$$
$$= \frac{330}{\pi}\cos\left(0.4 + \frac{2\pi}{55}(38)\right) \cong 3.02 \text{ ft}$$

$$y = R\sin(\theta_0 + \omega t)$$
$$= \frac{330}{\pi}\sin\left(0.4 + \frac{2\pi}{55}(38)\right) \cong -104.9989\ldots$$

Answer: $\boxed{(x(38), y(38)) \cong (3.02, -105) \text{ ft.}}$

   (b) (6 points) When does Maryam first pass the sitting duck, at the position indicated in the picture?

$$x = -60 = \frac{330}{\pi}\cos\left(0.4 + \frac{2\pi}{55}t\right)$$

$$\cos\left(0.4 + \frac{2\pi}{55}t\right) = \frac{-6\cancel{0}\pi}{33\cancel{0}} = \frac{-2\pi}{11}$$

Principal Sol: $0.4 + \frac{2\pi}{55}t = \cos^{-1}\left(\frac{-2\pi}{11}\right)$

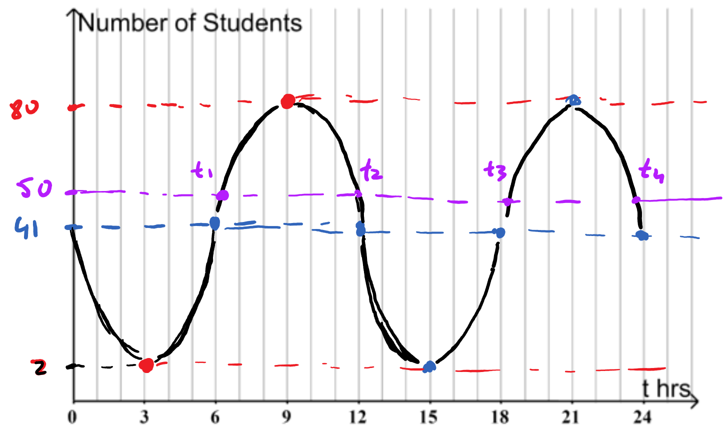
$$t = \frac{55}{2\pi}\left(\cos^{-1}\left(\frac{-2\pi}{11}\right) - 0.4\right)$$

$$\cong \boxed{15.57 \text{ sec.}}$$

[We need not search for additional solutions because, based on the picture, we can tell that Maryam will first reach the duck at a positive time t less than $\frac{1}{2}$ a lap, so this is it.]

2. (16 points) You are computing a sinusoidal model $S(t)$ for the number of Math 120 students who are studying at t hrs past midnight on March 15. Your data indicates that the highest number of students studying on that day was 80, first occuring at 9am. The lowest number of students studying was 2, first occurring at 3am.

   (a) Write down the function $S(t)$, in standard sinusoidal form, that models the number of students studying on March 15, at t hours past midnight. Also, sketch its graph.



$A = \frac{80-2}{2} = 39$

$D = \frac{80+2}{2} = 41$

$B = 2(9-3) = 12$ hrs

$C = 9 - \frac{B}{4} = 9-3 = 6 \ (\pm 12k)$

$$\boxed{S(t) = 39 \sin\left(\frac{2\pi}{12}(t-6)\right) + 41}$$

   (b) Find all times on March 15 when exactly 50 students were studying, according to this model.

$$39 \sin\left(\frac{2\pi}{12}(t-6)\right) + 41 = 50$$

$$39 \sin\left(\frac{\pi}{6}(t-6)\right) = 9$$

$$\sin\left(\frac{\pi}{6}(t-6)\right) = 9/39$$

Principal Sol.:

$\frac{\pi}{6}(t-6) = \sin^{-1}(9/39)$

$t = \frac{6}{\pi}\sin^{-1}(9/39) + 6 \cong \boxed{6.4447 = t_1}$

Symmetry Sol.:

$\pi/6 \ (t-6) = \pi - \sin^{-1}(9/39)$

$t = \frac{6}{\pi}\left(\pi - \sin^{-1}(9/39)\right) + 6 \cong \boxed{11.5552 \atop t_2}$

We get t_3 & t_4 by adding 1 period $= 12$ hrs to t_1 & t_2

Answer: At $\boxed{t \cong 6.44,\ 11.56,\ 18.44, \text{ and } 23.56 \text{ hrs}}$

6:26 AM, 11:33 AM, 6:26 PM, 11:33 PM.

3. Joe is a farmer and he knows that the number of pumpkins he'll sell is a linear function of the price per pumpkin. From previous years he knows that when he charges \$2 per pumpkin, he sells 2500 pumpkins. If he charges \$3 per pumpkin, he will sell 2000 pumpkins. Joe is trying to determine the price he should charge per pumpkin, to maximize his revenue (the total amount of money he gets for his pumpkins).

   (a) (6 points) Find the **linear** function $y = n(x)$ that computes the number of pumpkins sold, as a function of the price per pumpkin, x.

2 data points: $(x, n(x)) = (2, 2500)$ and $(3, 2000)$

slope: $m = \frac{2000-2500}{3-1} = -500$

point-slope: $y = -500(x-3) + 2000$

$= -500x + 1500 + 2000$

$= -500x + 3500$

$\boxed{n(x) = -500x + 3500}$

   (b) (8 points) What is the maximum possible revenue Joe can get from the sale of the pumpkins? What price should he charge per pumpkin, and how many pumpkins will Joe sell to get that maximal revenue?

The revenue: $r(x) = (\text{number sold}) \cdot (\text{price per pumpkin})$

$= n(x) \cdot x$

$r(x) = -500x^2 + 3500x$

This is a concave-down quadratic function $\Rightarrow$ max at vertex

$x = -b/2a = \frac{-3500}{2(-500)} = 3.5$ (\$/pumpkin)

So Joe should charge $\boxed{\$3.50/\text{pumpkin}}$

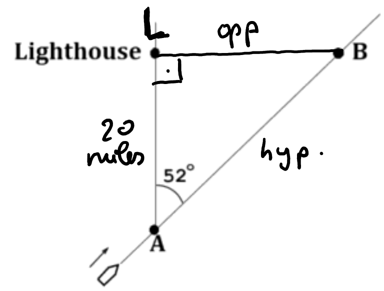
He'll sell $n(3.5) = -500(3.5) + 3500 = \boxed{1750 \text{ pumpkins}}$

and his max revenue will be

$r(x) = (1750)(3.5) = \boxed{\$6125}$

4. A ship is sailing in a straight line at constant speed in a northeasterly direction, as shown. At midnight the ship is at Point A, which is 20 miles due south of a lighthouse. At 3 AM, the ship gets to Point B, which is due east of the same lighthouse. The angle between the ship's path and the due north direction is 52 degrees, as shown. (Switch your calculator to degrees!)

(a) (5 points) What is the ship's speed?



$$\cos(52^\circ) = \frac{\text{adj}}{\text{hyp}} = \frac{20 \text{ miles}}{AB}$$

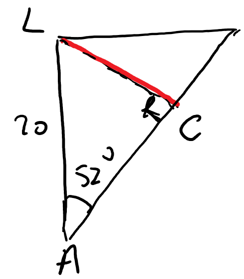
$$AB = 20/\cos(52^\circ) \cong 32.4854 \text{ miles}$$

$$\text{Speed} = \frac{AB}{3\text{hrs}} = \frac{32.4854}{3}$$

$$\cong \boxed{10.83 \text{ mph}}$$

(b) (5 points) What is the closest distance between the ship and the lighthouse?

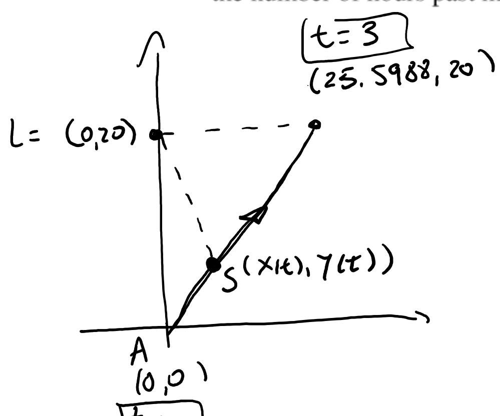
[There are many ways to compute this]



$$\sin(52^\circ) = \frac{LC}{20} \Rightarrow LC = 20 \sin 52^\circ$$

$$\cong \boxed{15.76 \text{ miles}}$$

(c) (6 points) Write the distance between the ship and the lighthouse as a function of t, where t is the number of hours past midnight.



$$\tan(52^\circ) = \frac{LB}{20} \Rightarrow LB = 20 \tan 52^\circ \cong 25.6$$

W/ origin at A:

ship moves from $(0,0)$ at $t=0$ to $(25.6, 20)$ at $t=3$

$$v_x = \frac{\Delta x}{\Delta t} = \frac{25.6}{3} \cong 8.53$$

$$v_y = \frac{\Delta y}{\Delta t} = \frac{20}{3} \cong 6.67$$

$$(x(t), y(t)) = (8.53t, 6.67t)$$

$$d(t) = \sqrt{(x(t)-0)^2 + (y(t)-20)^2} = \boxed{\sqrt{(8.53t)^2 + (6.67t-20)^2} \text{ miles}}$$

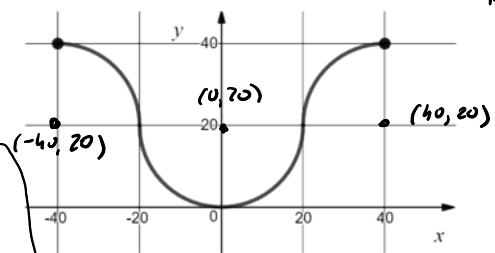
5. The vertical cross-section through a certain canal looks like in the picture below. It consists of two quarter circles and a half circle. All dimensions are in feet.

   (a) (6 points) Write down the multi-part function $y = f(x)$ corresponding to this canal, in the shown coordinate system. Make sure to include domains for each part.

$R = 20$ for all

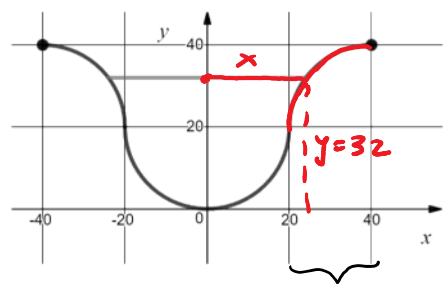
Top Semicircle: $y = k + \sqrt{R^2 - (x-h)^2}$

Bottom : $y = k - \sqrt{R^2 - (x-h)^2}$



$$\boxed{f(x) = \begin{cases} 20 + \sqrt{400 - (x+40)^2}, & -40 \le x \le -20 \\ 20 - \sqrt{400 - x^2}, & -20 \le x \le 20 \\ 20 + \sqrt{400 - (x-40)^2}, & 20 \le x \le 40. \end{cases}}$$

   (b) (8 points) Compute the horizontal width across this canal at a height of 32 feet above the x-axis.



$$y = 32 = \underbrace{20 + \sqrt{400 - (x-40)^2}}_{\text{equation for 3}^{rd}\text{ part}}$$

$$12 = \sqrt{400 - (x-40)^2}$$

$$144 = 400 - (x-40)^2$$

$$(x-40)^2 = 256$$

$$x - 40 = \pm 16 \Rightarrow x = 40 \pm 16$$

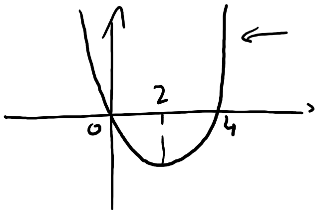
domain 20 to 40

$$x = 40 - 16 = 24$$

$$\text{width} = 2x = 2(24) = \boxed{48 \text{ feet}}$$

6. On this entire page, let: $f(x) = x^2 - 4x$, and $g(x) = 3^x$.

(a) (4 points) Find the range of f.



$\leftarrow f(x) = x^2 - 4x$

$\min y = y_{vertex} = f(2) = 2^2 - 4(2) = -4$

Range: $\boxed{-4 \leq y < \infty}$

(b) (6 points) Compute and simplify the following function compositions, as expressions in x.

- $f(x+3) = (x+3)^2 - 4(x+3) = x^2 + 6x + 9 - 4x - 12$
  $= \boxed{x^2 + 2x - 3}$.

- $f(g(2x)) = f(3^{2x}) = (3^{2x})^2 - 4(3^{2x}) = \boxed{3^{4x} - 4(3^{2x})}$.

(c) (6 points) Solve for x the equation $g(f(x)) = 5$. Show all steps.

$g(f(x)) = 3^{f(x)} = 3^{x^2-4x}$

So: $3^{x^2-4x} = 5$

Apply $\ln(\ldots) \Rightarrow$ $\ln(3^{x^2-4x}) = \ln(5)$

$(x^2 - 4x)\ln 3 = \ln 5$

$x^2 - 4x = \frac{\ln 5}{\ln 3}$

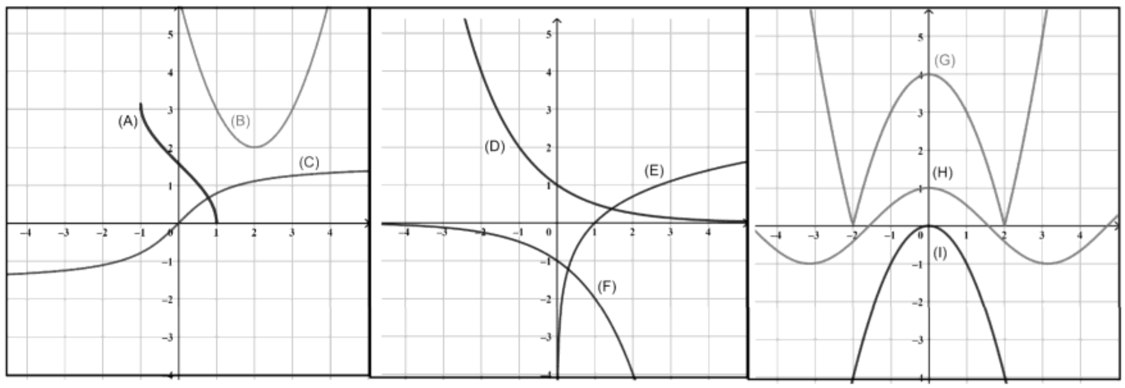
$x^2 - 4x - \frac{\ln 5}{\ln 3} = 0$

$x^2 - 4x - 1.464973\ldots = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{4 \pm \sqrt{16 - 4(1)(-1.464973\ldots)}}{2} = \frac{4 \pm 4.675456\ldots}{2}$

$\boxed{x \simeq 4.34 \ \& \ -0.34}$

7. (10 points) Consider the following graphs labeled (A)-(I):



For each of the functions listed below, circle ONE letter (A)-(I) for its corresponding graph above. If none of the graphs above match the function, then circle "None". Some graphs may match none of the functions. No need to justify or show any work.

(a) $y = 3x+7$ has graph: (A), (B), (C), (D), (E), (F), (G), (H), (I), or None of above

(b) $y = (x-2)^2+2$ has graph: (A), (B), (C), (D), (E), (F), (G), (H), (I), or None of above

(c) $y = \sqrt{x}$ has graph: (A), (B), (C), (D), (E), (F), (G), (H), (I), or None of above

(d) $y = |x^2-4|$ has graph: (A), (B), (C), (D), (E), (F), (G), (H), (I), or None of above

(e) $y = 2^{-x}$ has graph: (A), (B), (C), (D), (E), (F), (G), (H), (I), or None of above

(f) $y = -2^x$ has graph: (A), (B), (C), (D), (E), (F), (G), (H), (I), or None of above

(g) $y = \ln(x)$ has graph: (A), (B), (C), (D), (E), (F), (G), (H), (I), or None of above

(h) $y = \cos(x)$ has graph: (A), (B), (C), (D), (E), (F), (G), (H), (I), or None of above

(i) $y = \cos^{-1}(x) = \arccos(x)$ has graph: (A), (B), (C), (D), (E), (F), (G), (H), (I), or None of above

(j) $y = \tan^{-1}(x) = \arctan(x)$ has graph: (A), (B), (C), (D), (E), (F), (G), (H), (I), or None of above